Strange Equations Involving Functions 2022

Shortlisted Problems With Solutions

Problems

Algebra

A1 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$,

$$x^3 + f(x)f(y) = f(f(x^3) + f(xy)).$$

(*Emil Khalilov, Azerbaijan*)

A2 Find all functions $f : \mathbb{N} \to \mathbb{N}$ for which there exists $k \in \mathbb{N}$, such that for any $x, y \in \mathbb{N}$,

$$\frac{f(x+y) + f(x)}{ky + f(x)} = \frac{kx + f(y)}{f(x+y) + f(y)}.$$

(*Arkan Manva, India*)

A3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$,

$$f(xf(y)) + xf(x-y) = x^2 + f(2x).$$

(*hyay*)

A4 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for any $x, y \in \mathbb{N}$,

$$f^{x+f(y)-y}(xy) = xf(y) + 1.$$

(Note: for positive integers m, n, $f^0(n) = n$, $f^m(n)$ is f applied m times to n and $f^{-m}(n)$ is f^{-1} applied m times to n.)

(*Gabriel Goh, Singapore*)

A5 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$,

$$f(x + yf(x)) = f(xy + 1) + f(x - y).$$

(*Gabriel Goh, Singapore; Emil Khalilov, Azerbaijan*)

Combinatorics

C1 Define $[n] = \{1, 2, ..., n\}$ for all positive integers n. Find all n such that there exists a function $f : [n] \to [n]$ satisfying $|f(i) - if_i| = 1$ for all $i \in [n]$, where f_i denotes the number of $j \in [n]$ such that $f(j) = i$.

(Gabriel Goh, Singapore)

C2 Let $2^{[n]}$ denote the set of subsets of $[n] := \{1, 2, \dots, n\}$. Find all functions $f : 2^{[n]} \to 2^{[n]}$ which satisfy $|A \cap f(B)| = |B \cap f(A)|$ for all subsets A and B of $[n]$.

(Gabriel Goh, Singapore; Vlad Spătaru, Romania)

C3 Let π be a permutation of $[n] := \{1, 2, \dots, n\}$. Call a pair (i, j) an inversion if $i < j$ and $\pi(i) > \pi(j)$. Let I denote the number of inversions of π. Prove that

$$I \leq \sum_{k=1}^{n} |\, k - \pi(k)| \leq 2I$$

and find the equality cases.

(Arkan Manva, India)

Number Theory

N1 Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that for any positive integers m, n,

$$f(m+n) \mid f(m)+f(n) \text{ and } f(m)f(n) \mid f(mn).$$

(*Gabriel Goh, Singapore*)

N2 Let n be a fixed positive integer. Find all functions $f\colon \mathbb{N}\to\mathbb{N}$ such that for any $a, b \in \mathbb{N}$,

$$a+f(b) \mid af\left(a^{n-1}\right)+f(b)^n.$$

(*Aritra Mondal, India*)

N3 Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that for any integers x and y,

$$f(x)f(y)+f(xy)+x+y$$

is a prime number.

(*Dorlir Ahmeti, Kosovo; Gabriel Goh, Singapore*)

N4 Define $\mathbb{N}_0$ as the set of non-negative integers $\{0,1,2,\cdots\}$. Find all functions $f:\mathbb{N}_0\to\mathbb{N}_0$ such that

1. $f(0)=0$.

2. There exists a constant α such that $f(n^{2022}) \leq n^{2022}+\alpha$ for all $n\in\mathbb{N}_0$.

3. $af^b(a)+bf^c(b)+cf^a(c)$ is a perfect square for all $a,b,c\in\mathbb{N}_0$

(*Gabriel Goh, Singapore*)

N5 Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that for any $m,n\in\mathbb{N}$,

$$f^{f(m)}(n)|m+n+1.$$

(*Gabriel Goh, Singapore*)

Solutions

Algebra

A1 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that any real numbers x and y satisfy

$$x^3 + f(x)f(y) = f(f(x^3) + f(xy)).$$

(*Emil Khalilov, Azerbaijan*)

Solution. We claim that the only solutions are $f(x) = x$ and $f(x) = -x$. These can easily be verified to work. We now show these are the only solutions.

Let $P(x, y)$ denote the given assertion. If $f(0) \neq 0$ then, $P(0, x)$ gives us $f(x) = f(2f(0))/f(0)$, implying that f is constant. However, it is obvious that no constant function satisfies the equation.

Therefore, $f(0) = 0$. Using this in $P(\sqrt[3]{x}, 0)$ it follows that $f(f(x)) = x$, so f is an involution. Finally, by combining the latter with $P(1, f(x - f(1)))$ we get

$$1 + f(1)f(f(x - f(1))) = f(f(1) + f(f(x - f(1))))$$

which simplifies to $1 + f(1)(x - f(1)) = f(x)$, so f is a linear function.

Since $f(f(x)) = x$ it is then easy to see that $f(x) = x$ or $f(x) = -x$ are the only solutions.

A2 Find all functions $f : \mathbb{N} \to \mathbb{N}$ for which there exists $k \in \mathbb{N}$, such that for any $x, y \in \mathbb{N}$,

$$\frac{f(x+y)+f(x)}{ky+f(x)} = \frac{kx+f(y)}{f(x+y)+f(y)}.$$

(*Arkan Manva, India*)

Solution. We claim that the only functions that work are $f(x) = cx$ for some constant $c \in \mathbb{N}$. This works because we can take $k = 2c$ and the LHS and RHS are the same. We now show that these are the only functions.

Let $P(x, y)$ denote the assertion in question. First, note that $P(x, x)$ implies

$$\frac{f(2x)+f(x)}{kx+f(x)} = \frac{kx+f(x)}{f(2x)+f(x)}.$$

Thus, $f(2x) + f(x) = kx + f(x)$ so $f(2x) = kx$. Using the latter in $P(2x - 1, 1)$ we infer that

$$\frac{f(2x)+f(2x-1)}{k+f(2x-1)} = \frac{k(2x-1)+f(1)}{f(2x)+f(1)} \implies \frac{kx+f(2x-1)}{k+f(2x-1)} = \frac{k(2x-1)+f(1)}{kx+f(1)}.$$

Via cross multiplication and by taking common factors, we get that

$$(f(2x-1) - f(1) - k(x-1) = 0$$

for any $x > 1$. Note that this is obviously true for $x = 1$ as well. Finally, using the formulae we have deduced for $f(2x)$ and $f(2x - 1)$ in $P(2x, 1)$ we get that

$$\frac{f(1)+kx+kx}{k+kx} = \frac{2kx+f(1)}{2f(1)+kx}$$

which further simplifies to $2k^2x + kf(1) = 4kxf(1) + 2f(1)^2$, giving us $k = 2f(1)$.

Therefore, we get that $f(2x) = 2f(1)x = f(1)(2x)$ and $f(2x-1) = f(1)+2f(1)(x-1) = f(1)(2x-1)$, showing that $f(x)$ is of the form cx, as desired.

A3 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$,

$$f(xf(y)) + xf(x-y) = x^2 + f(2x).$$

(*hyay*)

Solution. The only solution is $f(x) = x + 1$. This can easily be verified to work. We show that it is the only solution.

Let $P(x, y)$ denote the assertion that $f(xf(y)) + xf(x-y) = x^2 + f(2x)$.

Case 1: f is injective.

$$P(f(x), f(x) - x) \text{ gives } f(f(x)f(f(x) - x)) = f(2f(x)) \implies f(x)f(f(x) - x) = 2f(x)$$

For $f(x) \neq 0$, $f(f(x) - x) = 2 \implies f(x) - x = c$ for some constant c.

If there exists a such that $f(a) = 0$, then $f(x) - x = c$ for all $x \neq a$ (by injectivity). Notice that we must have $a = -c$, otherwise taking $x = -c$ gives $f(-c) = 0$ $(= f(a))$, contradiction. Hence, $f(a) = a + c = 0$, and so $f(x) = x + c$ for all x (this conclusion still holds if no such a exist). Plugging it into $P(x, y)$ we get $f(x) = x + 1$ for all x.

Case 2: For some reals a, b, $f(a) = f(b)$.

Note that from $P(1, -1)$, we have $f(f(-1)) = 1$.

Let $a - b = c \neq 0$, then comparing $P(x + a, a)$ and $P(x + a, b)$ gives us $f(x) = f(x + c)$ for all $x \neq -a$. Take an arbitrary real number $t \notin \{-a + c, a\}$, then $f(t - c) = f(t)$, and by comparing $P(x + t, t)$ and $P(x + t, t - c)$ we also get $f(x) = f(x + c)$ for all $x \neq -t$. Hence this means $f(x) = f(x + c)$ actually holds for all x.

$P(x, f(-1))$ yields $f(x) + xf(x - f(-1)) = x^2 + f(2x)$ while $P(x + c, f(-1))$ gives $f(x) + (x + c)f(x - f(-1)) = (x + c)^2 + f(2x)$.

Comparing, $cf(x - f(-1)) = 2cx + c^2$. Thus, $f(x - f(-1)) = 2x + c$, which can be shown is never a solution.

A4 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for any $x, y \in \mathbb{N}$,

$$f^{x+f(y)-y}(xy) = xf(y) + 1.$$

(Note: for positive integers m, n, $f^0(n) = n$, $f^m(n)$ is f applied m times to n and $f^{-m}(n)$ is f^{-1} applied m times to n.)

(Gabriel Goh, Singapore)

Solution. We claim that the only solution is $f(x) \equiv x + 1$. This can easily be verified to work. We now show that this is the only solution.

Claim: For all positive integers n, $f(n) = n - 1$ or $f(n) \geq n + 1$.
Proof. Suppose on the contrary that $f(n) = n - c$ for some integer $c > 1$. Taking $P(c, n)$, we have $f^{c+f(n)-n}(cn) = cf(n) + 1 \implies cn = cf(n) + 1$. This means that $c|1$, contradiction. Furthermore, if $f(n) = n$, then $P(1, n) : f(n) = f(n) + 1$, contradiction. This proves the claim.

Case 1: For all positive integers n, $f(n) \geq n + 1$.

$P(1, y)$ yields $f^{1+f(y)-y}(y) = f(y) + 1$. However,

$$f^{1+f(y)-y}(y) \geq f^{f(y)-y}(y) + 1 \geq \ldots \geq f^1(y) + f(y) - y \geq f(y) + 1.$$

Hence, equality must hold everywhere and $f(x) = x + 1$ for all $x \in \mathbb{N}$, which works.

Case 2: There exists a positive integer a such that $f(a) = a - 1$.
Let t be the smallest number such that $f(t) = t - 1$. Obviously $t > 1$. From $P(x, t)$, $f^{x-1}(xt) = x(t - 1) + 1$. However,

$$f^{x-1}(xt) \geq f^{x-2}(xt) - 1 \geq \ldots \geq xt - (x - 1) = x(t - 1) + 1.$$

Hence, equality must hold everywhere and $f(xt - k) = xt - k - 1$ for all $x \geq 2$ and $0 \leq k \leq t - 1$.

This means that for all $n \geq t$, $f(n) = n - 1$. Let $f(t - 1) = t + c$ ($c \geq 1$). Note that by $P(t + c, 1)$, we have $f^{t+c+f(1)-1}(t + c) = (t + c)f(1) + 1$. The RHS cannot be part of the cycle of $t + c$ as it is more than $t + c$, a contradiction.

Thus, there is no solution in this case.

A5 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for any $x, y \in \mathbb{R}$,

$$f(x + yf(x)) = f(xy + 1) + f(x - y).$$

(*Gabriel Goh, Singapore; Emil Khalilov, Azerbaijan*)

Solution. The only solutions are $f(x) \equiv 0$ and $f(x) \equiv x - 1$. These can easily be verified to work, and we now prove they are the only solutions.

Let $P(x, y)$ denote the assertion that $f(x + yf(x)) = f(xy + 1) + f(x - y)$. The only constant solution is $f \equiv 0$, so assume f is non-constant from now on.

Claim 1: $f(1) = 0$, $f(1 + x) + f(1 - x) = 0$, $f(x - f(x)) = 0$ for all reals x.

Proof. Note that $P(0, 0)$ gives $f(0) = f(1) + f(0) \implies f(1) = 0$. Furthermore, from $P(1, y)$, we have $f(1 + y) + f(1 - y) = 0$. Lastly, $P(x, -1)$ yields $f(x - f(x)) = f(-x + 1) + f(x + 1) + 0$. This completes the claim.

Claim 2: f is not periodic.

Proof. Suppose on the contrary that $f(x + d) = f(x)$ for all $x \in \mathbb{R}$ and $d \neq 0$.
Notice that $P(x + d, y)$ implies $f(x + d + yf(x + d)) = f((x + d)y + 1) + f(x + d - y)$. Using the periodicity of f, $f(x + yf(x)) = f(xy + xd + 1) + f(x - y)$. By comparing this with $P(x, y)$, we get $f(xy + 1) = f(xy + xd + 1)$.

Taking $y \to \frac{dy}{x}$, $x \to \frac{x}{d}$, $f(y + 1) = f(y + x + 1)$, which means f is constant, contradiction.

Claim 3: $f(c) = 0 \implies c = 1$.

Proof. Suppose $f(c) = 0$. Then, $P(c, y)$ implies $0 = f(cy + 1) + f(c - y)$. Taking y to be $-y$, we get $0 = f(-cy + 1) = f(c + y)$. However, by claim 1, we know that $f(cy + 1) + f(-cy + 1) = 0$, hence $f(c - y) + f(c + y) = 0$. Note that $f(c + y) = -f(c - y) = -(-f(2 - c + y)) = f(y + (2 - c))$. Hence, f is periodic with period $2(c - 1)$. By claim 2, we must have $c = 1$.

Finally, combining $f(x - f(x)) = 0$ with Claim 3, we have $f(x) = x - 1$ for all reals x, which is a solution.

Combinatorics

C1 Define $[n] = \{1, 2, ..., n\}$ for all positive integers n. Find all n such that there exists a function $f : [n] \to [n]$ satisfying $|f(i) - if_i| = 1$ for all $i \in [n]$, where f_i denotes the number of $j \in [n]$ such that $f(j) = i$.

(*Gabriel Goh, Singapore*)

Solution. We claim that only even n satisfy the statement. Begin by noticing that indeed, by taking $f(2k) = 2k - 1$ and $f(2k - 1) = 2k$ for all $1 \leq k \leq n/2$, we have $f_i = 1$ for all i and $|f(i) - if_i| = |f(i) - i| = 1$ for all i.

We will proceed to show that no such function exists for odd n. By a simple double-counting one can observe that $f(1) + f(2) + \cdots f(n) = 1 \cdot f_1 + 2 \cdot f_2 + \cdots n \cdot f_n$. Moreover, note that

$$|f(i) - if_i| = 1 \implies f(i) \equiv if_i + 1 \pmod 2$$

and by summing the latter over all $1 \leq i \leq n$ and using the aforementioned identity, we get that

$$\sum_{i=1}^{n} f(i) \equiv n + \sum_{i=1}^{n} if_i = n + \sum_{i=1}^{n} f(i) \pmod 2$$

so n must be even, giving us a contradiction. Therefore, the answer is all even n.

C2 Let $2^{[n]}$ denote the set of subsets of $[n] := \{1, 2, \ldots, n\}$. Find all functions $f : 2^{[n]} \to 2^{[n]}$ which satisfy $|A \cap f(B)| = |B \cap f(A)|$ for all subsets A and B of $[n]$.

(*Gabriel Goh, Singapore; Vlad Spătaru, Romania*)

Solution. A function f satisfies the given condition if and only if $f(\emptyset) = \emptyset$, there exists some subset $K \subseteq [n]$ upon which f acts as an involution, $f(\{k\}) = \emptyset$ for all $k \in [n] \setminus K$, and for all $A \subseteq 2^{[n]}$ we have

$$f(A) = \left(\bigcup_{a \in A} f(\{a\}) \right).$$

It is trivial to observe that $f(\emptyset) = \emptyset$. We first prove necessity, beginning with two claims.

Claim 1. For all $A \in 2^{[n]}$ we have $|f^2(A)| = |f(A)|$ and $f^2(A) \subseteq A$.

Proof. Note that $P(A, f(A))$ gives us $|f(A)| = |A \cap f^2(A)|$. Since $|X \cap Y| \leq \min(|X|, |Y|)$ it follows that $|f(A)| \leq |A|$ and $|f(A)| \leq |f^2(A)|$. By substituting $A \to f(A)$ into the former, we get $|f^2(A)| \leq |f(A)|$ and by combining this with the latter, we get $|f^2(A)| = |f(A)|$.

Furthermore, since $|f^2(A)| = |f(A)|$ and $|f(A)| = |A \cap f^2(A)|$ then $f^2(A) \subseteq A$. □

Now consider the following set: $K := \{k \in [n] : f(\{k\}) \neq \emptyset\}$. Assume that $k \in K$. By claim 1, note that $|f^2(\{k\})| = |f(\{k\})| > 0$. However, we also know that $f^2(\{k\}) \subseteq \{k\}$ and therefore, $f^2(\{k\}) = \{k\}$. It follows that f acts as an involution (and thus bijection) on K.

Claim 2. For all $A \in 2^{[n]}$ and $k \in [n]$, if $\{k\} \subseteq A$ then $f(\{k\}) \subseteq f(A)$.

Proof. If $k \notin K$ then $f(\{k\}) = \emptyset \subseteq A$ so the claim holds. Otherwise, $P(A, f(\{k\}))$ yields

$$|A \cap f^2(\{k\})| = |f(A) \cap f(\{k\})|.$$

However, as discussed above, $f^2(\{k\}) = \{k\} \subseteq A$ so $|f(A) \cap f(\{k\})| = 1$ but since $|f(\{k\})| = |f^2(\{k\})| = |\{k\}| = 1$, we then get the desired $f(\{k\}) \subseteq f(A)$. □

Using claim 2 we can define a new function $h : 2^{[n]} \to 2^{[n]}$ as such:

$$h(A) := f(A) \setminus \left(\bigcup_{a \in A} f(\{a\}) \right).$$

Assume that for some $a, b \in [n]$ we have $f(\{a\}) = \{b\}$. Since $\{b\} \neq \emptyset$ then $a \in K$ and because f is bijective over K then $b \in K$ as well. Thus, since f is an involution over K we get that

$$f(\{a\}) = \{b\} \iff f^2(\{a\}) = f(\{b\}) \iff \{a\} = f(\{b\}).$$

Since $f(\{a\}) = \{b\}$ if and only if $\{a\} = f(\{b\})$ we can easily infer that

$$\left| A \cap \left(\bigcup_{b \in B} f(\{b\}) \right) \right| = \left| B \cap \left(\bigcup_{a \in A} f(\{a\}) \right) \right| \tag{1}$$

or, in other words, $|A \cap (f(B) \setminus h(B))| = |B \cap (f(A) \setminus h(A))|$ but by using $P(A, B)$ we get that $|A \cap h(B)| = |B \cap h(A)|$. Therefore, h also satisfies the given condition.

Furthermore, note that $h(\{t\}) = f(\{t\}) \setminus f(\{t\}) = \emptyset$ for all $t \in [n]$. Now, if for some A we have $h(A) \neq \emptyset$ then there exists a such that $a \in h(A)$. However, by plugging in $B = \{a\}$ in $|A \cap h(B)| = |B \cap h(A)|$ we have

$$1 = |h(A) \cap \{a\}| = |h(\{a\}) \cap A| = |\emptyset \cap A| = 0$$

which is a contradiction. Therefore, $h(A) = \emptyset$ for all $A \in 2^{[n]}$. By plugging this in the definition of h it follows that for all $A \in 2^{[n]}$ we have

$$f(A) = \left(\bigcup_{a \in A} f(\{a\}) \right)$$

where $f(\emptyset) = \emptyset$, f is an involution on some $K \subseteq [n]$, and sends all elements in $[n] \setminus K$ in $\emptyset$, which are the desired criteria. Finally, note that sufficiency was proven alongside equation (1).

Remark. After proving claim 2, an alternate finish goes as such:

Claim 3. For all $A \in 2^{[n]}$ and $k \in [n]$, if $\{k\} \subseteq f(A)$ then $|f(\{k\})| = 1$ and $f(\{k\}) \subseteq A$.

Proof. Note that $P(A, \{k\})$ gives us $1 = |\{k\} \cap f(A)| = |f(\{k\}) \cap A|$. Recall that $|f(\{k\})| \in \{0, 1\}$. If $|f(\{k\})| = 0$, or in other words $f(\{k\}) = \emptyset$, then $|f(\{k\}) \cap A| = 0 \neq 1$.

Therefore, $|f(\{k\})| = 1$ and since $1 = |f(\{k\}) \cap A|$ it follows that $f(\{k\}) \subseteq A$. □

Using claim 3 we can infer that for all A, no $k \in [n] \setminus K$ is in $f(A)$, since it would contradict $|f(\{k\})| = 1$, as $f(\{k\}) = \emptyset$ for all $k \in [n] \setminus K$. Thus, for any A, if $\{k\} \subseteq f(A)$, then $k \in K$.

Now, assume that for some $A \in 2^{[n]}$ and $k \in K$, $\{k\} \in f(A)$. By claim 3 we know that $f(\{k\}) \subseteq A$ so there exists $i \in A$ such that $f(\{k\}) = \{i\}$. Since $k \in K$ and f is bijective over K, then $i \in K$ as well. Furthermore, f is an involution, so

$$f(\{k\}) = \{i\} \iff f^2(\{k\}) = f(\{i\}) \iff k = f(i).$$

Thus, all elements of $f(A)$ are equal to $f(\{i\})$ for some $i \in A \cap K$. This, combined with claim 2 implies the fact that

$$f(A) = \left(\bigcup_{a \in A} f(\{a\}) \right).$$

It only remains to show that f satisfies the hypothesis and all the conditions we enforced, like we did in the previous solution.

C3 Let π be a permutation of $[n] := \{1, 2, \ldots, n\}$. Call a pair (i, j) an inversion if $i < j$ and $\pi(i) > \pi(j)$. Let I denote the number of inversions of π. Prove that

$$I \leq \sum_{k=1}^{n} |k - \pi(k)| \leq 2I$$

and find the equality cases.

(*Arkan Manva, India*)

Solution. We work on the LHS and RHS separately.

We first show that $\sum\limits_{k=1}^{n} |k - \pi(k)| \leq 2I$.

Proof. Consider some number a between 1 and n. Suppose there are $x - 1$ $(x > 0)$ numbers $j_1, \cdots j_{x-1}$ such that $j_i < a$ and $\pi(j_i) > \pi(a)$. Obviously, $x \leq a, \pi(a)$. Then, there are $a - x$ numbers which are $< a$ and have a π value of $> \pi(a)$, which means they form an inversion with a. Similarly, there are $\pi(a) - x$ numbers which are $> a$ and have a π value of $< \pi(a)$, also forming an inversion with a.

Thus, the number of inversions containing a are $\pi(a) + a - 2x$. We wish to show this is $\geq |a - \pi(a)|$.

If $\pi(a) \geq a$, then it suffices to show $\pi(a) + a - 2x \geq \pi(a) - a \iff a \geq x$, which is true. Equality holds when $x = a$, which means $\pi(a)$ is larger than $\pi(1), \pi(2), \cdots, \pi(a-1)$.

If $\pi(a) \leq a$, then it suffices to show $\pi(a) + a - 2x \geq a - \pi(a) \iff \pi(a) \geq x$, which is true. Equality holds when $x = \pi(a)$, which means $\pi(a)$ is smaller than $\pi(a+1), \pi(a+2), \cdots \pi(n)$.

We now find the equality case. An intuitive way to combine both cases is that a cannot be inversions with numbers both $< a$ and $> a$. This is also sufficient as with this condition, one of the above inequalities must be true. Hence, equality holds when there does not exists d, e, f such that $1 \leq d < e < f \leq n$ and $\pi(d) > \pi(e) > \pi(f)$.

We now show that $\sum\limits_{k=1}^{n} |k - \pi(k)| \geq I$. Let $K = \sum\limits_{k=1}^{n} |k - \pi(k)|$.

Lemma. If τ is a bijection from $A = \{a_1 < \cdots < a_m\}$ to $B = \{b_1 < \cdots < b_m\}$ then there exists j such that the number of inversions that j is in is in is at most $|a_j - b_{\pi(j)}|$.

Proof. If $a_m \geq b_m$ then $a_m - b_{\pi(m)} \geq b_m - b_{\pi(m)} \geq m - \pi(m)$. However, the number of inversions m is in is precisely $m - \pi(m)$. If $b_m \geq a_m$ we can pick j such that $\pi(j) = m$ to get the same result. Furthermore, equality holds if and only if $a_m = b_m$. Otherwise, we can show n is the desired index.

Back to the original problem. We consider the following process: remove a number k such that $|k - \pi(k)|$ is greater than the number of inversions containing k. k exists by the lemma. Eventually, both K and I will reach 0, and since K always decreases more than I, $K \geq I$. From the lemma, equality holds when $\max(a_i) = \max(b_j)$, which means $\pi(n) = n$. After removing n, $\pi(n-1) = n-1$, etc. Hence, the only equality case is $\pi = id$.

Number Theory

N1 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for any positive integers m, n,

$$f(m+n) \mid f(m)+f(n) \text{ and } f(m)f(n) \mid f(mn).$$

(Gabriel Goh, Singapore)

Solution. The only solution is $\boxed{f(m) \equiv 1}$ and $\boxed{f(m) \equiv m}$. We claim that they are the only solutions.

Setting $m = n = 1$ in $f(m)f(n) \mid f(mn)$, we get $f(1) = 1$.
Setting $n = 1$ in $f(m+n) \mid f(m) + f(n)$, we get $f(m+1) \mid f(m)+1$. Thus, $f(m+1) \leq f(m)+1$.
Setting $m = n = 1$ in $f(m+n) \mid f(m) + f(n)$, we get $f(2) \mid 2$.

Case 1. $f(2) = 2$.
We show by induction that $f(2^k) = 2^k$. Suppose it is true for k. Consider $k+1$.

From the second equation, $f(2^k)f(2) \mid f(2^{k+1})$ or $2^{k+1} \mid f(2^{k+1})$. From the first equation, $f(2^{k+1}) = f(2^k + 2^k) \mid 2f(2^k) = 2^{k+1}$. Hence, this gives $f(2^{k+1}) = 2^{k+1}$, which completes the induction.

Note that $2^{k+1} = f(2^{k+1}) \leq f(2^{k+1}-1)+1 \leq f(2^{k+1}-2)+2 \leq \cdots \leq f(2^k+1)+(2^k-1) \leq f(2^k)+2^k = 2^{k+1}$.

Thus, equality must hold everywhere and $f(2^k + m) = 2^{k+1} - 2^k + m = 2^k + m$ for all $k \in \mathbb{N}$ and $1 \leq m \leq 2^k - 1$. This means $f(n) = n$ for all $n \in \mathbb{N}$ which obviously works.

Case 2. $f(2) = 1$.
Suppose c be the smallest number such that $f(c) \neq 1$. Then, $c > 2$. Since $f(c) \mid f(c-1)+f(1) = 2$, we must have $f(c) = 2$.

Claim: $f(n) = 2$ for all $c \mid n$ and $f(n) = 1$ otherwise.
We use strong induction. The claim holds for $n = 1, 2, ..., c$.

For any other number m,

- if $m = ck$ for some k, then we have $f(ck) \mid f(1)+f(ck-1) = 2$ and $f(c)f(k) \mid f(ck) \implies 2 \mid f(ck)$. This implies $f(ck) = 2$.
- if $m = ck + t$ for some k and $2 \leq t \leq c-1$, then $f(ck+t) \mid f(t) + f(ck) = 3$ and $f(ck+t) \mid f(1) + f(ck+t-1) = 2$, forcing $f(ck+t) = 1$.
- if $m = ck+1$ for some k, then $f(ck+1) \mid f(ck)+f(1) = 3$ and $f(ck+1) \mid f(ck-1)+f(2) = 2$, forcing $f(ck+1) = 1$.

This proves our claim. However, taking $m = n = c$ in the second equation, $4 \mid 2$, which is a contradiction.

Hence, the only solution in this case occurs when c does not exist, i.e. $f(n) = 1 \forall n \in \mathbb{N}$, which obviously works.

N2 Let n be a fixed positive integer. Find all functions $f\colon \mathbb{N} \to \mathbb{N}$ such that for any $a, b \in \mathbb{N}$,

$$a + f(b) \mid af\left(a^{n-1}\right) + f(b)^n.$$

(*Aritra Mondal, India*)

Solution. We claim that the only the following functions work:

- $f \equiv 1$.
- If $n = 2$, any constant function works.
- If n is odd, any function such that $f(a^{n-1}) = a^{n-1}$ works.

It is easy to show these functions work. We now show that they are the only solutions.

Let $P(a, b)$ be the assertion that $a + f(b) \mid af(a^{n-1}) + f(b)^n$.

Case 1: f is bounded.

Suppose $f(a) \leq N$ for all a. Then, exists $c \in \mathbb{N}$ such that $f(t^{n-1}) = c$ for infinitely many t.

From $P(t, b)$, $t + f(b) \mid tc + f(b)^n \implies t + f(b) \mid f(b)^n - cf(b)$. Since the LHS is unbounded, $f(b)^n = cf(b)$ for all b.

We first deal with the case $n > 1$. This means f is constant. Let $f(a) \equiv c$. We thus have $a + c \mid ac + c^n \implies a + c \mid c^n - c^2$, and thus $c^n - c^2 = 0$. Hence, if $n = 2$ then c can be anything, else $c = 1$.

If $n = 1$, then we get $c = 1$, and any function such that $f(1) = 1$ works (for $n = 1$). This is covered in the third case in the solution set above.

Case 2: f is unbounded.

If n is even, then $a + f(b) \mid a^n - f(b)^n \implies a + f(b) \mid af(a^{n-1}) + a^n$. Fixing a, we have $a + f(b) \leq af(a^{n-1}) + a^n$, which means $f(b)$ is bounded, a contradiction.

Hence, f is odd. Thus, $a + f(b) \mid a^n + f(b)^n \implies a + f(b) \mid af(a^{n-1}) - a^n$. Since f is unbounded, the RHS is zero, so $f(a^{n-1}) = a^{n-1}$. Any such function works.

N3 Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that for any integers x and y,

$$f(x)f(y) + f(xy) + x + y$$

is a prime number.

(Dorlir Ahmeti, Kosovo; Gabriel Goh, Singapore)

Solution. We claim that the only functions satisfying the condition is $f(x) \equiv 1 - x \forall x \in \mathbb{Z}$. This works because the expression is always 2, which is a prime. We now show that this is the only function.

Let $P(x, y)$ denote the expression $f(x)f(y) + f(xy) + x + y$. Also, define $p(x) = x + f(x) + 1$.

Lemma 1: $f(0) = 1$ and $f(1) = 0$. *Proof.* From $P(1, 1)$, $f(1)^2 + f(1) + 2$ is prime. This is always even, hence must be equal to 2, which means $f(1) = 0$ or -1.

If $f(1) = -1$, $P(1, y)$ gives that $-f(y) + f(y) + y + 1 = y + 1$ is always prime, a contradiction. Thus, $f(1) = 0$.

$P(0, 0) = f(0)^2 + f(0)$ is a prime. Similarly, this is always even, so $f(0)^2 + f(0) = 2$. Thus, $f(0) = 1$ or -2. From $P(1, 0)$, $f(0) + 1$ is a prime, which means $f(0) = 1$. This proves the claim.

As a consequence, $P(x, 1) : f(x) + x + 1 = p(x)$ is a prime. $(\star)$

Case 1: $f(2) = -1$.

First, we find $f(-1)$ and $f(-2)$. $P(-1, -2)$ shows that $f(-1)f(-2) - 4$ is prime. At the same time, note that by $\star$, $f(-2) - 1$ and $f(-1)$ are primes. Let $f(-2) - 1 = a$ and $f(-1) = b$. Then, $b(a+1) - 4$ is prime. If a is odd, then $b(a+1) - 4 \geq 2(4) - 4 = 4$, but yet it is even, a contradiction. Hence $a = 2$ and so $f(-2) = 3$.

$P(-1, 2) : f(-2) - f(-1) + 1 = 4 - f(-1)$ is prime. Since $f(-1)$ is prime, it must be equal to 2. Thus, $f(-1) = 2$ and $f(-2) = 3$.

Note that $P(x, -2)$ gives $3f(x) + f(-2x) + x - 2 = 3(f(x) + x + 1) + (f(-2x) - 2x + 1) - 6 = 3p(x) + p(-2x) - 6$ is prime.

Similarly, $P(2x, -1)$ gives $2f(2x) + f(-2x) + 2x - 1 = 2(f(2x) + 2x + 1) + (f(-2x) - 2x + 1) - 4 = 2p(2x) + p(-2x) - 4$ is prime.

Lastly, $P(x, 2)$ gives $f(2x) - f(x) + x + 2 = (f(2x) + 2x + 1) - (f(x) + x + 1) + 2 = p(2x) - p(x) + 2$ is prime.

In summary, $p(2x) - p(x) + 2$, $3p(x) + p(-2x) - 6$, $2p(2x) + p(-2x) - 4$ are primes. We claim that $p(x) = 2$.

If $p(2x) \neq p(x)$, then since $p(2x) - p(x) + 2$ is a prime, one of the $p(x), p(2x)$ must be even. If $p(2x) = 2$ then $p(x) = 2$, contradicting $p(2x) \neq p(x)$. Thus, $p(x)$ is even so $p(x) = 2$. If $p(2x) = p(x) \neq 2$, then $3p(x) + p(-2x) - 6$ and $2p(x) + p(-2x) - 4$ are both primes. However, they differ by $p - 2$, which is odd. Hence, the smaller one must be 2, i.e. $2p(x) + p(-2x) - 4 = 2$. This is only possible when $p(x) = 2$. Thus, $f(x) + x + 1 = 2 \implies f(x) = 1 - x$ for all x, which is a solution.

Case 2: $f(2) \neq -1$.

$P(2,2) : f(2)^2 + f(4) + 4$ is prime. Note that by $\star$, $f(2) + 3$ and $f(4) + 5$ are primes.

If $f(2) = 0$ then $P(2,8)$ implies $f(16) + 10$ is a prime, but $f(16) + 17$ is also a prime. Since they have different parity, $f(16)+10 = 2 \implies f(16)+17 = 9$, which is not a prime, a contradiction.

Hence $f(2) > 0$. Let $f(2) = a - 3$. Then, $(a-3)^2 + f(4) + 4 = f(4) + a^2 - 6a + 13$ is prime. Note that a is odd and ≥ 5) (since $a = f(2) + 3$ is a prime > 3), so $f(4) = a^2 - 6a + 13$ is even and ≥ 8. At the same time, $f(4)+5$ is prime, and since 5 is odd, $f(4)+5$ must be 2. Thus, $f(4) = -3$.

$P(-1,-4)$ gives $f(-1)f(-4) - 8$ is a prime. However, $f(-1)$ and $f(-4) - 3$ are both primes. Hence, if $f(-4)$ is even then $f(-1)f(-4) - 8 \geq 2 \times 6 - 8 \geq 4$, yet it is even, a contradiction. Hence $f(-4)$ is odd so $f(-4) - 3$ is even, i.e. $f(-4) = 5$.

At the same time, $P(-1,4)$ gives $-3f(-1) + f(-4) + 3 = 8 - 3f(-1)$ is prime, so $f(-1) = 2$ (because $f(-1)$ is prime as well).

From $P(8,-1)$, we get that $2f(8) + f(-8) + 1 = 2p(8) + p(-8) - 4$ is prime.

Also, $P(4,2)$ implies $-3f(2) + f(8) + 6 = p(8) - 3p(2) + 6$ is prime.

If $p(2)$ is even then $f(2) + 3 = 2$, so $f(2) = -1$, a contradiction.

Hence, $p(2)$ is odd. If $p(8)$ is even, then $2p(8) + p(-8) - 4$ is even, so must be 2. Then, this means $p(8) = 2$, so $p(8) - 3p(2) + 6 = 8 - 3p(2)$, so $p(2) = 2$ which is even, a contradiction. Lastly, if $p(8)$ is odd, then $p(8) - 3p(2) + 6$ is even, hence must be 2. Thus, $p(8) = 3p(2) - 4$.

From $P(8,-1)$, $2p(8) + p(-8) - 4 = 6p(2) + p(-8) - 12$ is a prime. $P(-4,2)$ also gives $5f(2) + f(-8) - 2 = 5p(2) + p(-8) - 10$ is a prime. These differ by $p(2) - 2$, which is odd, so the smaller one must be 2. Note $f(2) + 3$ is prime, and since $f(2)$ is odd and more than -1, $f(2) \geq 2$. Thus, $5p(2) + p(-8) - 10$ is smaller, and thus must be 2.

This forces $p(2) = 2$ anyways, which is a contradiction.

Hence, there is no solution in this case.

N4 Define $\mathbb{N}_0$ as the set of non-negative integers $\{0, 1, 2, \cdots\}$. Find all functions $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that

1. $f(0) = 0$.

2. There exists a constant α such that $f(n^{2022}) \leq n^{2022} + \alpha$ for all $n \in \mathbb{N}_0$.

3. $af^b(a) + bf^c(b) + cf^a(c)$ is a perfect square for all $a, b, c \in \mathbb{N}_0$

(*Gabriel Goh, Singapore*)

Solution. The only functions that work are:

1. $f(a) \equiv 0$.
2. $f(0) = 0$, $f(a > 0) = a + 2$.

It is easy to show these work (however, note that the verification is non-trivial but due to the length of the solution they will not be shown). We now show that these are the only solutions.

Claim 1: If f is not constantly 0, then f is injective at 0.
Proof. Suppose $f(t) = 0$ for some $t > 0$.

$P(t, b > 0, 1) = 0 + bf(b) + f^t(1)$ and $P(0, b, 1) = bf(b) + 1$.

Note that if $f^k(1) = 1$, then $P(k, 0, 1) = k^2 + 1$, so $k = 0$. Hence, $f^t(1) \neq 1$.

If there are infinitely many b such that $f(b) > 0$, then taking a sufficiently large b satisfying this property, we must have that $f^t(1) - 1$ is the difference of 2 arbitrarily large perfect squares, which is not possible.

Hence, $f(b) = 0$ for all $b \geq M$ for some constant M. Thus, there exists a constant L such that $f(a) < L$ for all $a \in \mathbb{N}_0$.

$P(1, l, 0) = f^l(1) + l^2$. Taking a sufficiently large $l > M, L$, $f^l(1) = 0$ and $f(l) = 0$. Hence, $P(l, b > 0, 1) = 0 + bf(b) + f^l(1) \implies bf(b)$ is a perfect square. However, we have $bf(b) + 1$ is a perfect square as well, hence $bf(b) = 0 \implies f(b) = 0 \forall b > 0$.

Thus, $f \equiv 0$. This concludes the claim.

Claim 2: $f(a) = a + 2$ for infinitely many a.
Proof. $P(0, t, 1) = tf(t) + 1$.

Take $t = p^{2022}$ for some large prime p and suppose $tf(t) + 1 = k^2$. Thus, $tf(t) = (k - 1)(k + 1)$. Note $k > 1$ by claim 1. Also, t has to divide either $k - 1$ or $k + 1$, so $t \leq k + 1$.

If $(k + 1) \geq 2t$, then, $f(t) = \frac{(k-1)(k+1)}{t} \geq 2(k - 1) \geq 2(t - 1 - 1)$. However, $2(t - 2) > t + \alpha$ for sufficiently large p. Hence, for sufficiently large p, $k + 1 < 2t$. This means $k = t + 1$ or $k = t - 1$, i,e, $f(t) = t - 2$ or $t + 2$.

If $f(t) = t - 2$, $P(1, 1, t) = f(1) + f^t(1) + t^2 - 2t$ is a perfect square $\geq t^2$. $P(1, t, 0) = f^t(1) + t^2$ is a perfect square $\geq t^2$. Hence, their difference, $2t - f(1)$, is either 0 or $\geq 2t + 1$. Obviously, it has to be 0, hence $f(1) = 2t$. However, taking p arbitrarily large, this is not possible.

To conclude, for all sufficiently large p, $f(p^{2022}) = p^{2022} + 2$.

Let the set $\mathbb{A} = \{a_1, a_2, ...\}$ be the set of positive integers such that $f(a_i) = a_i + 2$ and a_i is the $2022th$ power of a prime. From claim 2, we know that $|A|$ is infinite.

Claim 3: $f(a > 0) = a + 2$.
Proof. From now on, assume $x, y \in \mathbb{A}$, where x, y are sufficiently large. [sl] means that we are using the fact that x/y is sufficiently large.

$P(1, 1, x) = f(1) + f^x(1) + x^2 + 2x$ and $P(1, p, 0) : f^x(1) + x^2$. Since both are perfect squares more than x^2 with a difference of $2x + f(1)$, we can assume they are consecutive squares [sl]. Thus, $f^x(1) + x^2 = (x + \frac{f(1)-1}{2})^2$. Let $k = \frac{f(1)-1}{2} \implies f^x(1) = 2xk + k^2$.

For any b, $P(1, b, x) = f^b(1) + bf^x(b) + x^2 + 2x$ and $P(0, b, x) = bf^x(b) + x^2$. Since these are perfect squares $> x^2$ with difference $f^b(1) + 2x$, they are consecutive squares [sl].

This means $bf^x(b) + x^2 = (x + \frac{f^b(1)-1}{2})^2 \implies bf^x(b) = 2(\frac{f^b(1)-1}{2})x + (\frac{f^b(1)-1}{2})^2$ $(\star)$.

Taking $b = y$ in $(\star)$, $yf^x(y) = 2(\frac{f^y(1)-1}{2})x + (\frac{f^y(1)-1}{2})^2 = (\frac{f^y(1)-1}{2})(2x + \frac{f^y(1)-1}{2}) = (yk + \frac{k^2-1}{2})(2x + yk + \frac{k^2-1}{2})$.

Note that y divides the LHS. Hence, for any y [sl], we can take a small $x << \sqrt[2022]{y}$, so that $\sqrt[2022]{y} > 2x + \frac{k^2-1}{2}$. This forces $y \mid \frac{k^2-1}{2}$ for all sufficiently large y, so $k = 1$.

This gives $f^x(1) = 2x + 1$. Hence, $xf^y(x) = 2xy + x^2$.

Now, for any a, let $k_a = \frac{f^a(1)-1}{2}$ (which must be integer by $(\star)$). Also, by $(\star)$, $af^x(a) = 2k_a x + k_a^2$. $P(a, x, y) = af^x(a) + xf^y(x) + +yf^a(y) = 2k_a x + k_a^2 + 2xy + x^2 + yf^a(y)$. This is equivalent to $(x + y + k_a)^2 + (-2yk_a + yf^a(y) - y^2)$. Hence, $yf^a(y) = 2yk_a + y^2$ by taking a sufficiently large x.

Finally, for any a, b, $P(a, b, y) = af^b(a) + bf^y(b) + yf^a(y) = af^b(a) + 2k_b y + k_b^2 + 2yk_a + y^2$. This is equivalent to $(y + k_a + k_b)^2 + (af^b(a) - 2k_a k_b - k_a^2)$. Similarly, $af^b(a) = 2k_a k_b + k_a^2$.

Let the assertion above be $Q(a, b)$.
$Q(3, 2) : 3f^2(3) = 2k_3 k_2 + k_3^2$. $Q(1, 3) : f^3(1) = 2k_3 + 1$ (because $k_1 = 1$). Since $f(1) = 3$, $f^3(1) = f^2(3)$. Thus, $6k_3 + 3 = 2k_3 k_2 + k_3^2$. This means $k_3 | 3$.

Obviously, if $k_3 = 1$ then $f^3(1) = 3 \implies f(f(3)) = 3$. Then, $Q(3, x \in \mathbb{A})$ is bounded on the LHS but not on the RHS, contradiction.
Hence, $k_3 = 3 \implies 18 + 3 = 6k_2 + 9 \implies k_2 = 2$. $Q(3, b) : 3f^b(3) = 6k_b + 9$. Since $f^b(3) = f^{b+1}(1) = 2k_{b+1} + 1$, this means $6k_{b+1} + 3 = 6k_b + 9$. Hence, $k_{b+1} = k_b + 1$. By induction, $k_b = b$.

Lastly, $Q(a > 0, 1) : af(a) = 2a + a^2 \implies f(a) = a + 2$. This concludes the proof.

N5 Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for any $m, n \in \mathbb{N}$,

$$f^{f(m)}(n) \mid m + n + 1.$$

(*Gabriel Goh, Singapore*)

Solution. The only functions that work are:

1. $f(n) \equiv 1 \ \forall n \in \mathbb{N}$,
2. $f(2n-1) = 2$ and $f(2n) = 1 \ \forall n \in \mathbb{N}$,
3. $f(2n-1) = 2$, $f(2) = 1$ and $f(2n+2) =$ is any divisor of $2n+5$ greater than $1 \ \forall n \in \mathbb{N}$ and
4. $f(n) \equiv n + 1 \ \forall n \in \mathbb{N}$.

These can easily be verified to work. We now show that these are the only solutions.

Case 1: There exist positive integers $a > b$ such that $f(a) = f(b) = 1$.
$P(a, n) : f(n) \mid a + n + 1$, $P(b, n) : f(n) \mid b + n + 1$. Thus, $f(n) \mid a - b$. For any prime $p > a + 1$, $P(a, p - a - 1) : f(p - a - 1) \mid p$. However, since $gcd(p, a - b) = 1$, $f(p - a - 1) = 1$.

Thus, $P(p - a - 1, n) : f(n) \mid n + p - a \implies f(n) \mid (n + p - a) - (n + a + 1) \implies f(n) \mid p - 2a - 1$. By Dirichlet Theorem, we can take $p \equiv 1 \pmod{a - b}$, which implies $f(n) \mid 1 - 2a - 1$.

Thus, $f(n) \mid 2a$. This means $f(n) \mid p - 1$. By Dirichlet Theorem, we can take $p \equiv -1 \pmod{a - b}$. which implies $f(n) \mid -1 - 1 \implies f(n) \mid 2$.

Thus, $f(n) = 1$ or 2 for each n. If there exists a, b such that $f(a) = f(b) = 1$ and $a \equiv b + 1 \pmod 2$, then since $f(n) \mid a - b$ and $a - b$ is odd, we must have $\boxed{f(n) \equiv 1 \forall \mathbb{N}}$.

Else, if $a \equiv 1 \pmod 2$, then $f(2k) = 2 \forall k \in \mathbb{N}$. $P(2, 2) : f^{f(2)}(2) \mid 5 \implies f(f(2)) \mid 5 \implies 2 \mid 5$, a contradiction.

Lastly, if $a \equiv 0 \pmod 2$, $f(2k - 1) = 2 \forall k \in \mathbb{N}$. If $f(2) = 2$, $P(1, 1) : f(f(1)) \mid 3 \implies 2 \mid 3$, a contradiction. Hence $f(2) = 1$. If there exists c such that $f(2c) = 2$, then $P(2c, 2) : f(f(2)) \mid 2c + 2 + 1 \implies f(1) \mid 2c + 2 + 1 \implies 2 \mid 2c + 2 + 1$, a contradiction. Hence, $f(2k) = 1 \forall k \in \mathbb{N}$. This gives the second solution, $\boxed{f(2n - 1) = 2 \text{ and } f(2n) = 2 \forall n \in \mathbb{N}}$

Case 2: There exists a unique positive integer c such that $f(c) = 1$.
Lemma. There does not exist a, b such that $f^a(b) = b$ and the cycle of b does not contain 1.
Proof: Suppose otherwise. Let $x_1 = f(b)$, $x_2 = f(f(b))$, ..., $x_a = b$. Let $P = \prod x_i$. $P(mP - b, b)$: $f^{f(mP-b)}(b) \mid mP + 1$. However, $f^{f(mP-b)}(b) \mid P$ as well, so $f^{f(mP-b)}(b) = 1$, a contradiction.

$P(c, n) : f(n) \mid n + c + 1$. For any prime $p > 2c + 1$, $f(p - c - 1) \mid p$. Since $p - c - 1 > c$, $f(p - c - 1) = p$. $P(p - c - 1, c) : f^{p-1}(1) \mid p$.

Subcase 1: There exists t such that $f^t(1) = 1$.
Let k be the minimal integer such that $f^k(1) = 1$. k must exist since t is such an integer.

Subcase 1.1: $k = 1$.
Then, $f(1) = 1$ and $c = 1 \implies f(n) \mid n + 2$. Thus, $f(3) \mid 5 \implies f(3) = 5$ and similarly $f(5) = 7$. Note $f(7) \mid 9$ If $f(7) = 3$ then $3, 5, 7$ is a cycle, contradiction. Thus, $f(7) = 9$. Similarly, $f(9) = 11$, $f(11) = 13$. Hence $P(3, 3) : f^5(3) \mid 7$, but $f^5(3) = 13$, a contradiction.

Subcase 1.2: $k = 2$.
This means that $c = f(1)$. Thus, $f(1) \mid 1 + c + 1 \implies f(1) \mid 2$. If $f(1) = 1$, then $k = 1$, contradiction. Thus $f(1) = 2$ and $f(2) = 1$.

Hence, $P(2, n) : f(n) \mid n + 3$ and in particular $f(3) \mid 6$. If $f(3) = 3$ then this contradicts the lemma. If $f(3) = 6$, then $f(6) \mid 9 \implies f(6) = 9$. Then $P(1, 3) : f^2(3) \mid 5 \implies 9 \mid 5$, contradiction.

If $f(3) = 2$, then $P(1, n) : f(f(n)) \mid n + 2$ and $P(3, n) : f(f(n)) \mid n + 4$. This means $f(f(n)) \mid 2$. Furthermore, taking n odd, since $n + 2$ is odd, we must have $f(f(n)) = 1$. By injectivity at 1, $f(n) = 2$, so $f(2n - 1) = 2 \forall n \in \mathbb{N}$.

Take $n > 1$. Note that $f(2n)$ divides $2n + 3$. Any such function work, giving the third solution: $\boxed{f(2n-1) = 2, f(2) = 1 \text{ and } f(2n+2) = \text{ is any divisor of } 2n+5 \text{ greater than } 1\ \forall n \in \mathbb{N}}$.

Subcase 1.3: $k > 2$.
Take some sufficiently large prime $q \equiv -1 \pmod{k}$, which is larger than any number in the cycle starting from 1. Note that $f^{q-1}(1) \mid q$. Since it can't be q, it must be 1. Thus, $f^{q-1}(1) = 1$. This means that $k \mid q - 1 \implies k \mid 2$, so this case is not possible.

Subcase 2: There is no cycle containing 1. (By the lemma, there is no cycle at all.)
Take prime $p > 2c + 1$. From above, $f^{p-1}(1) \mid p$. Hence, $f^{p-1}(1) = p$.
$P(p - c - 1, 1) : f^p(1) \mid p - c + 1$. Hence, $f(p) \mid p - c + 1$. Since $f(p) \mid p + c + 1$ as well, this means $f(p) \mid 2c$. Hence, there must exist p, q such that $f(p) = f(q) \implies f^p(1) = f^q(1)$, which is a contradiction since there cannot be any cycles.

Case 3: There does not exist any positive integer c such that $f(c) = 1$.

Lemma 3.1. f is injective.

Proof: If $f(a) = f(b) = c$ and $a > b$, then $P(a, n), P(b, n) : f^c(n) \mid a + n + 1$ and $f^c(n) \mid b + n + 1$. This means $f^c(n) \mid a - b$. Taking $n = k(a - b) - a$ for sufficiently large n, $f^c(n) \mid a + k(a - b) - a + 1$ so $f^c(n) | k(a - b) + 1$. This means $f^c(n) = 1$, a contradiction.

Lemma 3.2. f does not contain a cycle.

Proof. $P(p - 2, 1), P(p - n - 1, n)$ for sufficiently large p gives $f^{f(p-2)}(1) = p = f^{f(p-n-1)}(n)$.

Since there does not exist $f(c) = 1$, $f(p - 2) > f(p - n - 1) \implies f^{f(p-2)-f(p-n-1)}(1) = n$. Thus, we can reach any number through fs from 1. This shows that f does not contain a cycle.

Lemma 3.3 $f^{f(m)}(1) = m + 2$ for sufficiently large m.

Proof. Consider the set $\mathbb{S} = \{f(1), f^{f(1)}(1), \cdots, f^{f(m)}(1)\}$.

Since f has no cycles and $\{1, f(1), \cdots, f(m)\}$ are distinct, therefore all elements in $\mathbb{S}$ are distinct. At the same time, $f^{f(m)}(1) \leq m + 2$. Hence, for any $m > f(1)$, all elements of $\mathbb{S}$ are between 2 and $m+2$, inclusive. This means $\mathbb{S} = \{2, 3, \cdots m+2\}$ for sufficiently large m. By considering $m+1$ and subtracting the respective sets, we get that $f^{f(m+1)}(1) = m+3$, finishing the proof of this claim.

Lemma 3.4 $f^{f(m)}(2) = m + 3$ for sufficiently large m.

Proof. Similar to lemma 3.3, showing that the set $\{f(1), f(2), f^{f(1)}(2), \cdots, f^{f(m)}(2)\}$ are distinct.

Lemma 3.5 $f(m) = m + 1$ for sufficiently large m.

Proof. From the two lemmas above, $f^{f(m+1)}(1) = f^{f(m)}(2)$ for sufficiently large m. By injectivity and the fact that there are no cycles, we know that $f(m + 1) - f(m)$ is a constant for sufficiently large m. However, by lemma 3.2, f is surjective at every point except 1, hence $f(m+1) = f(m)+1$ for sufficiently large m.

Finally we are ready to finish the proof (wow good job you made it here).

Taking a sufficiently large n, we have $f^{f(m)}(n) = f(m)+n | m+n+1$, and so $f(m)+n \mid m+1-f(m)$. This gives the 4th solution $\boxed{f(n) \equiv n + 1 \forall n \in \mathbb{N}}$.