50 Functional Equations for the Seasoned Contestant

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1 Introduction

Complaints against Functional Equations are not unusual, and comments such as "every FE is boring nowadays"[1] or "very very extremely ridiculous standard FE"[2] have emerged whenever a "classic" FE appears in a reputable contest. While it is unfortunately true that FEs require a lot more "bashing" and less "insight" then most other olympiad subtopics, every once in a while an FE materialises which challenges (or even shatters) this notion. I usually categorise these into two types:

1. FEs that look standard, yet have some surprising twist or some idea that requires way more intuition and creativity than the norm. Here one thinks of IMO'17/2 as the most well-known example.

2. FEs whose problem statement just challenge convention, and the best IMO example here would be IMO'22/2 (which, once you ignore the fact that it's a problem 2, is a pretty neat problem). Usually these problems entail the 'E' of 'FE' not being a strict *equation*, but more of a freestyle 'condition'.

These constitute my favourite types of FEs , and they are a refreshing break from the usual plug-and-chug questions. Now that I have finished my contestant journey, I took some time to compile them. I have ordered them roughly by difficulty, according to my highly subjective opinions. Please note that this is definitely not meant to be a problem set for learning FEs - there are handouts for that[3,4]. I tried to include the source wherever possible, and a link if the source is sufficiently obscure. Most of the problems are very recent, both because FE proposals are getting quirkier and because these are the problems I grew up with. I also included several "classic" questions for completeness[5]. Enjoy!

*You can find me as *gghx* on both AoPS and discord

[1] https://artofproblemsolving.com/community/c6h1954632p13509989

[2] https://artofproblemsolving.com/community/c6h1876068p12745214

[3] https://artofproblemsolving.com/community/c6h411461p2308754

[4] https://artofproblemsolving.com/community/c6h1592427p9873821

[5] And I may have sneaked in a few of my originals :p

2 Problems

1. (EMC(J) 2014[6]) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that
   - $f(mn) = f(m)f(n)$ for all positive integers m, n.
   - There are infinitely many n such that $f(1), f(2), \cdots, f(n)$ is a permutation of $1, 2, \cdots, n$.

2. (IMOC 2023[7]) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers n, there exists an unique positive integer k satisfying
$$f^k(n) \leq n + k + 1.$$

3. (NICE MO 2021[8]) For each prime p, let $\mathbb{S}_p = \{1, 2, \ldots, p-1\}$. Find all primes p for which there exists a function $f \colon \mathbb{S}_p \to \mathbb{S}_p$ such that
$$n \cdot f(n) \cdot f(f(n)) - 1 \text{ is a multiple of } p$$

4. (ELMO SL 2019[9]) Let $f : \mathbb{N} \to \mathbb{N}$. Show that $f(m) + n \mid f(n) + m$ for all positive integers $m \leq n$ if and only if $f(m) + n \mid f(n) + m$ for all positive integers $m \geq n$.

5. (FEOO SL 2020[10]) Let k be a fixed positive integer. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for any distinct positive integers $a_1, a_2, ..., a_k$, there exist a permutation of its $b_1, b_2, ..., b_k$ such that,
$$\frac{f(a_1)}{b_1} + \frac{f(a_2)}{b_2} + ... + \frac{f(a_k)}{b_k}$$
is a positive integer.

6. (Taiwan TST R1 2022) Find all $f : \mathbb{Z} \to \mathbb{Z}$ such that
$$f\left(\left\lfloor \frac{f(x)+f(y)}{2} \right\rfloor\right) + f(x) = f(f(y)) + \left\lfloor \frac{f(x)+f(y)}{2} \right\rfloor$$
holds for all $x, y \in \mathbb{Z}$.

7. (Japan MO 2020) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that
$$m^2 + f(n)^2 + (m - f(n))^2 \geq f(m)^2 + n^2$$
for all pairs of positive integers (m, n).

8. (IMO 2022) Let $\mathbb{R}^+$ denote the set of positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for each $x \in \mathbb{R}^+$, there is exactly one $y \in \mathbb{R}^+$ satisfying
$$xf(y) + yf(x) \leq 2$$

9. (Israel TST 2019) Find all functions $f : \mathbb{R} \to \mathbb{R}_{\geq 0}$ such that for any reals x, y,
$$f(f(x) - y^2) + f(2xy) = f(x^2 + y^2).$$

[6] https://artofproblemsolving.com/community/c6h2759347p24127944
[7] https://artofproblemsolving.com/community/c6h3152411p28637527
[8] https://artofproblemsolving.com/community/c1806461h2516372p21323591
[9] https://artofproblemsolving.com/community/c6h1864642p12623620
[10] https://artofproblemsolving.com/community/c6h2129239p15548089

10. (Japan MO 2023) Let m be a positive integer. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for any $n \in \mathbb{N}$, there are exactly $f(n)$ positive integers k satisfying $f(k) \leq f(n+1) + m$.

11. (InfinityDots MO 2018[11]) Determine all bijections $f : \mathbb{Z} \to \mathbb{Z}$ satisfying

$$f^{f(m+n)}(mn) = f(m)f(n)$$

for all integers m, n.

12. (Taiwan TST 2022[12]) Let $\mathcal{X}$ be the collection of all non-empty subsets (not necessarily finite) of the positive integer set $\mathbb{N}$. Determine all functions $f : \mathcal{X} \to \mathbb{R}^+$ satisfying the following properties:

   (a) For all $S, T \in \mathcal{X}$ with $S \subseteq T$, there holds $f(T) \leq f(S)$.

   (b) For all $S, T \in \mathcal{X}$, there hold

   $$f(S) + f(T) \leq f(S+T), \quad f(S)f(T) = f(S \cdot T),$$

   where $S + T = \{s + t \mid s \in S, t \in T\}$ and $S \cdot T = \{s \cdot t \mid s \in S, t \in T\}$.

13. (MOMO 2020[13]) Suppose that there exist a nonempty set $X \subset \mathbb{R}$ and a function $f : X \to X$ satisfying

$$f(x) + y \in X \text{ if and only if } x \neq y$$

for every $x, y \in X$. Prove that $f(x) + x$ is constant while x varies on X.

14. (Japan MO 2009) Find all functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for any non-negative reals x, y,

$$f(x^2) + f(y) = f(x^2 + y + xf(4y)).$$

15. (China TST 2018) Functions $f, g : \mathbb{Z} \to \mathbb{Z}$ satisfy

$$f(g(x) + y) = g(f(y) + x)$$

for any integers x, y. If the range of f is finite, prove that g is periodic.

16. (MEMO 2020) Let $\mathbb{N}$ be the set of positive integers. Determine all positive integers k for which there exist functions $f : \mathbb{N} \to \mathbb{N}$ and $g : \mathbb{N} \to \mathbb{N}$ such that g assumes infinitely many values and such that

$$f^{g(n)}(n) = f(n) + k$$

holds for every positive integer n.

17. (Canada MO 2021) A function f from the positive integers to the positive integers is called *Canadian* if it satisfies

$$\gcd\left(f(f(x)), f(x+y)\right) = \gcd(x, y)$$

for all pairs of positive integers x and y. Find all positive integers m such that $f(m) = m$ for all Canadian functions f.

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[12] https://artofproblemsolving.com/community/c6h2835273p25098318
[13] https://artofproblemsolving.com/community/c6h1984152p13801379

18. (ISL 2017) Let S be a finite set, and let $\mathcal{A}$ be the set of all functions from S to S. Let f be an element of $\mathcal{A}$, and let $T = f(S)$ be the image of S under f. Suppose that $f \circ g \circ f \neq g \circ f \circ g$ for every g in $\mathcal{A}$ with $g \neq f$. Show that $f(T) = T$.

19. (SEIF 2022[14]) Let $2^{[n]}$ denote the set of subsets of $[n] := \{1, 2, \ldots, n\}$. Find all functions $f : 2^{[n]} \to 2^{[n]}$ which satisfy
$$|A \cap f(B)| = |B \cap f(A)|$$
for all subsets A and B of $[n]$.

20. (USEMO 2020[15]) A function f from the set of positive real numbers to itself satisfies
$$f(x + f(y) + xy) = xf(y) + f(x + y)$$
for all positive real numbers x and y. Prove that $f(x) = x$ for all positive real numbers x.

21. (AoPS user TLP.39[16]) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that
$$\sum_{i=1}^{n^2} f(i) = n^2 f(n)$$
for all $n \in \mathbb{N}$.

22. (Singapore MO 2022) Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ satisfying
$$m!! + n!! \mid f(m)!! + f(n)!!$$
for each $m, n \in \mathbb{Z}^+$, where $n!! = (n!)!$ for all $n \in \mathbb{Z}^+$.

23. (Summer MO 2020[17]) Let $p > 2$ be a fixed prime number. Find all functions $f : \mathbb{Z} \to \mathbb{Z}_p$, where the $\mathbb{Z}_p$ denotes the set $\{0, 1, \ldots, p - 1\}$, such that p divides $f(f(n)) - f(n + 1) + 1$ and $f(n + p) = f(n)$ for all integers n.

24. (Brazil MO 2019) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for any positive reals x, y,
$$f(xy + f(x)) = f(f(x)f(y)) + x.$$

25. (Balkan MO 2022) Find all functions $f : (0, \infty) \to (0, \infty)$ such that
$$f(y(f(x))^3 + x) = x^3 f(y) + f(x)$$
for all $x, y > 0$.

26. (IMOC 2019[18]) Find all functions $f : \mathbb{N} \to \mathbb{N}$ so that
$$f^{2f(b)}(2a) = f(f(a + b)) + a + b$$
holds for all positive integers a, b.

[14] https://artofproblemsolving.com/community/c6h2800059p24662655
[15] https://artofproblemsolving.com/community/c6h2318789p18486884
[16] https://artofproblemsolving.com/community/c6h2772397p24313301
[17] https://artofproblemsolving.com/community/c6h2251630p17350753
[18] https://artofproblemsolving.com/community/c6h2651731p22956810

27. (Japan MO 2020) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that there exists a positive constant c satisfying
$$\gcd(f(m) + n, f(n) + m) > c(m + n)$$
for all positive integers m, n.

28. (HMIC 2023[19]) Suppose $f : \mathbb{N} \to \mathbb{N}$ is a function such that for any positive integers m, n,
$$f(m + n) \mid f(m)f(n) - 1.$$
Show that for all sufficiently large positive integers n, $f(n) = 1$.

29. (KoMaL A.825[20] modified) Find all functions $f : \mathbb{Z}^+ \to \mathbb{R}^+$ such that for any positive integer m, n, $f(mn) = f(m)f(n)$ and $\lim\limits_{n\to\infty} \frac{f(n+1)}{f(n)} = 1$.

30. (ISL 2022) Let $\mathbb{R}$ be the set of real numbers. We denote by $\mathcal{F}$ the set of all functions $f \colon \mathbb{R} \to \mathbb{R}$ such that
$$f(x + f(y)) = f(x) + f(y)$$
for every $x, y \in \mathbb{R}$ Find all rational numbers q such that for every function $f \in \mathcal{F}$, there exists some $z \in \mathbb{R}$ satisfying $f(z) = qz$.

31. (ISL 2015) Let $\mathbb{Z}_{>0}$ denote the set of positive integers. Consider a function $f : \mathbb{Z}_{>0} \to \mathbb{Z}_{>0}$. For any $m, n \in \mathbb{Z}_{>0}$ we write $f^n(m) = \underbrace{f(f(\dots f}_{n}(m)\dots))$. Suppose that f has the following two properties:
    (a) if $m, n \in \mathbb{Z}_{>0}$, then $\frac{f^n(m) - m}{n} \in \mathbb{Z}_{>0}$;
    (b) The set $\mathbb{Z}_{>0} \setminus \{f(n) \mid n \in \mathbb{Z}_{>0}\}$ is finite.

    Prove that the sequence $f(1) - 1, f(2) - 2, f(3) - 3, \dots$ is periodic.

32. (GAMO 2022[21]) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for any positive integers x, y,
$$f^{f(x)+y}(y) = f(x + y) + y.$$

33. (APMO 2021) Determine all Functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(f(a) - b) + bf(2a)$ is a perfect square for all integers a and b.

34. (ISL 2020) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ satisfying
$$f^{a^2+b^2}(a + b) = af(a) + bf(b)$$
for all integers a and b.

35. (Israel TST 2023) Find all functions $f : \mathbb{Z} \to \mathbb{Z}_{>0}$ for which
$$f(x + f(y))^2 + f(y + f(x))^2 = f(f(x) + f(y))^2 + 1$$
holds for any $x, y \in \mathbb{Z}$.

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[20] https://artofproblemsolving.com/community/c6h2842657p25183546
[21] https://artofproblemsolving.com/community/c6h2845189p25210898

36. (China MO 2021) Find $f : \mathbb{N} \to \mathbb{N}$, such that for any $x, y \in \mathbb{Z}_+$,

$$f(f(x) + y) \mid x + f(y).$$

37. (USATSTST 2022) Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f\colon \mathbb{N} \to \mathbb{Z}$ such that

$$\left\lfloor \frac{f(mn)}{n} \right\rfloor = f(m)$$

for all positive integers m, n.

38. (EMC 2020) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$xf(x + y) + f(xf(y) + 1) = f(xf(x))$$

for all $x, y \in \mathbb{R}^+$.

39. (IMOC 2022[22]) Let the set of all bijective functions taking positive integers to positive integers be $\mathcal{B}$. Find all functions $\mathbf{F} : \mathcal{B} \to \mathbb{R}$ such that

$$(\mathbf{F}(p) + \mathbf{F}(q))^2 = \mathbf{F}(p \circ p) + \mathbf{F}(p \circ q) + \mathbf{F}(q \circ p) + \mathbf{F}(q \circ q)$$

for all $p, q \in \mathcal{B}$.

40. (RMM 2012) Each positive integer is coloured red or blue. A function f from the set of positive integers to itself has the following two properties:

- if $x \leq y$, then $f(x) \leq f(y)$; and
- if x, y and z are (not necessarily distinct) positive integers of the same colour and $x+y = z$, then $f(x) + f(y) = f(z)$.

Prove that there exists a positive number a such that $f(x) \leq ax$ for all positive integers x.

41. (IMO 2017) Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that, for any real numbers x and y,

$$f(f(x)f(y)) + f(x + y) = f(xy).$$

42. (IMOC 2022[23]) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x + y)f(f(x)) = f(1 + yf(x))$$

for all $x, y \in \mathbb{R}^+$.

43. (Macedonia TST 2022[24]) We consider all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n) + n) = n$ and $f(a + b - 1) \leq f(a) + f(b)$ for all positive integers a, b, n. Prove that there are at most two values for $f(2022)$.

44. (Korea Winter Program 2019[25]) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that if a, b, c are the length sides of a triangle, and r is the radius of its incircle, then $f(a), f(b), f(c)$ also form a triangle where its radius of the incircle is $f(r)$.

[22] https://artofproblemsolving.com/community/c6h2918274p26069694
[23] https://artofproblemsolving.com/community/c6h2918270p26069674
[24] https://artofproblemsolving.com/community/c6h2849151p25254155
[25] https://artofproblemsolving.com/community/c6h1766848p11573447

45. (IRN-SGP-TWN 2023[26]) Find all $f : \mathbb{Z}[x] \to \mathbb{Z}[x]$ such that for any integer polynomials P, Q and integer r we have

$$P(r) \mid Q(r) \iff f_P(r) \mid f_Q(r).$$

(We define $a|b$ if and only if $b = za$ for some integer z. In particular, $0|0$.)

Remark: Take note this is not division in the polynomial sense. f_P *is shortform for* $f(P)$, *because* $f(P)(r)$ *just dosen't look right.*

46. (SEIF 2022[27]) Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for any $m, n \in \mathbb{N}$,

$$f^{f(m)}(n) \mid m + n + 1.$$

47. (ISL 2015) For every positive integer n with prime factorization $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, define

$$\mho(n) = \sum_{i:\ p_i > 10^{100}} \alpha_i.$$

That is, $\mho(n)$ is the number of prime factors of n greater than 10^{100}, counted with multiplicity.

Find all strictly increasing functions $f : \mathbb{Z} \to \mathbb{Z}$ such that

$$\mho(f(a) - f(b)) \le \mho(a - b) \quad \text{for all integers } a \text{ and } b \text{ with } a > b.$$

48. (USATST 2014) Find all functions $f : \mathbb{N} \to \mathbb{Z}$ such that for any positive integers m, n,

$$(m - n)(f(m) - f(n))$$

is always the square of an integer.

49. (USAMO 2022) Let $\mathbb{R}_{>0}$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ such that for all $x, y \in \mathbb{R}_{>0}$ we have

$$f(x) = f(f(f(x)) + y) + f(xf(y))f(x + y).$$

50. (GIMO 2021[28]) Determine all functions f mapping positive reals to positive reals such that

$$f(x)f(x + 2f(y)) = xf(x + y) + f(x)f(y)$$

for all positive reals x, y.

[26] https://artofproblemsolving.com/community/c6h3112293p28153628
[27] https://artofproblemsolving.com/community/c6h2800033p24662520
[28] https://artofproblemsolving.com/community/c6h2595544p22384013

3 Bonus

(ELMO 2019 P6) Carl chooses a *functional expression** E which is a finite nonempty string formed from a set $x_1, x_2, \ldots$ of variables and applications of a function f, together with addition, subtraction, multiplication (but not division), and fixed real constants. He then considers the equation $E = 0$, and lets S denote the set of functions $f\colon \mathbb{R} \to \mathbb{R}$ such that the equation holds for any choices of real numbers $x_1, x_2, \ldots$. (For example, if Carl chooses the functional equation

$$f(2f(x_1) + x_2) - 2f(x_1) - x_2 = 0,$$

then S consists of one function, the identity function.

(a) Let X denote the set of functions with domain $\mathbb{R}$ and image exactly $\mathbb{Z}$. Show that Carl can choose his functional equation such that S is nonempty but $S \subseteq X$.

(b) Can Carl choose his functional equation such that $|S| = 1$ and $S \subseteq X$?

*These can be defined formally in the following way: the set of functional expressions is the minimal one (by inclusion) such that (i) any fixed real constant is a functional expression, (ii) for any positive integer i, the variable x_i is a functional expression, and (iii) if V and W are functional expressions, then so are $f(V)$, $V + W$, $V - W$, and $V \cdot W$.